Quantum Mechanics I

Week 2 (Solutions)

Spring Semester 2025

1 Matrix Diagonalization

Find the eigenvalues and the normalized eigenvectors of the following matrices.

The objective of this exercise is to determine the eigenvalues and eigenvectors of the following square matrices. For a matrix A, the eigenvalues are obtained by solving the characteristic equation

$$\det(A - \lambda \mathbb{1}) = 0\,. \tag{1.1}$$

Once this is achieved, we then determine the corresponding eigenvectors. For the eigenvalues α_i, we solve the equation

$$Av_i = \alpha_i v_i\,, \tag{1.2}$$

with the condition that the eigenvector v_i must be normalized, i.e. $v_i^\dagger v_i = 1$. For a 2×2 matrix, like A_1, the above takes the form of

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \alpha_i \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad |x|^2 + |y|^2 = 1\,, \tag{1.3}$$

where $(x_i \quad y_i)^T$ corresponds to the eigenvector v_i.

We provide the eigenvalues and normalized eigenvectors for the following matrices:

(a) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The eigenvalues of A_1 are equal to $\lambda_1 = 1$ and $\lambda_2 = -1$. The corresponding eigenvectors are given by $v_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $v_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$.

(b) $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The eigenvalues of A_2 are equal to $\lambda_1 = 1$ and $\lambda_2 = -1$. The corresponding eigenvectors are given by $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}^T$ and $v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}^T$.

(c) $A_3 = \begin{pmatrix} 0 & 2i \\ -2i & 2 \end{pmatrix}$

The eigenvalues of A_3 are equal to $\lambda_\pm = \pm 2$. The corresponding eigenvectors are given by $v_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mp i \end{pmatrix}^T$.

(d) $A_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

The eigenvalues of A_4 are equal to $\lambda_1 = 0, \lambda_2 = +1, \lambda_3 = -1$. Their respective eigenvectors are: $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 0 & 1\end{pmatrix}^T, v_2 = \frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}^T, v_3 = \frac{1}{2}\begin{pmatrix}1 & -\sqrt{2} & 1\end{pmatrix}^T$.

Remark: The following matrices share the form with physical observables that we frequently encounter in quantum mechanics. Keep these examples in mind for the future. Try to identify them during the course.

2 Hermitian Operators

In the lecture we have seen that Hermitian operators play an important role in quantum mechanics, as they are associated with physical observables. In the following, you will prove some important properties of these operators.

(a) Show that for two Hermitian operators we have that

$$\left(\hat{A}\hat{B}\right)^\dagger = \hat{B}^\dagger \hat{A}^\dagger. \tag{2.1}$$

Hint: Consider the matrix elements of $\hat{A}\hat{B}$.

We consider the matrix elements of the matrix $\left(\hat{A}\hat{B}\right)^\dagger$, and the action of the conjugate transpose operation on these elements:

$$\text{LHS} = ((AB)^\dagger)_{ij} = ((AB)_{ji})^* .$$

Then, using the definition of the matrix-matrix multiplication $(AB)_{ij} = \sum_k A_{ik}B_{kj}$, we may write:

$$\text{LHS} = \left(\sum_k A_{jk}B_{ki}\right)^* = \sum_k B^*_{ki}A^*_{jk} .$$

In the latter, we change the order of the matrix elements since they are simply numbers. Using the property of the conjugate transpose operation, we write:

$$\text{LHS} = \sum_k B^*_{ki}A^*_{jk} = \sum_k (B^\dagger)_{ik}(A^\dagger)_{kj} = (B^\dagger A^\dagger)_{ij} ,$$

where in the last equality we used the definition of the matrix-matrix multiplication.

(b) Show that the following operators are Hermitian:

$$\hat{A}^\dagger\hat{A}, \quad \hat{A}\hat{A}^\dagger, \quad \frac{\hat{A}+\hat{A}^\dagger}{2}, \quad \frac{\hat{A}-\hat{A}^\dagger}{2i} . \tag{2.2}$$

Using the result from Question (a) and $(A^\dagger)^\dagger = A$, we find:

$$
\begin{aligned}
(A^\dagger A)^\dagger &= A^\dagger (A^\dagger)^\dagger = A^\dagger A\,,\\
(AA^\dagger)^\dagger &= (A^\dagger)^\dagger A^\dagger = A^\dagger A\,,\\
\left(\frac{A+A^\dagger}{2}\right)^\dagger &= \frac{A^\dagger + A}{2}\,,\\
\left(\frac{A-A^\dagger}{2i}\right)^\dagger &= \frac{A^\dagger - (A^\dagger)^\dagger}{-2i} = \frac{A-A^\dagger}{2i}\,.
\end{aligned}
$$

(c) Show that if $\hat{H}$ is Hermitian, then $\hat{A}\hat{H}\hat{A}^\dagger$ and $\hat{A}^\dagger\hat{H}\hat{A}$ are also Hermitian.

Using the relations from Question (b), we find:

$$
\begin{aligned}
(\hat{A}\hat{H}\hat{A}^\dagger)^\dagger &= (\hat{H}\hat{A}^\dagger)^\dagger \hat{A}^\dagger = (\hat{A}^\dagger)^\dagger \hat{H}^\dagger \hat{A}^\dagger = \hat{A}\hat{H}\hat{A}^\dagger\,,\\
(\hat{A}^\dagger\hat{H}\hat{A})^\dagger &= (\hat{H}\hat{A})^\dagger (\hat{A}^\dagger)^\dagger = \hat{A}^\dagger\hat{H}^\dagger\hat{A}\,.
\end{aligned}
$$

Using that $\hat{H} = \hat{H}^\dagger$ we can conclude that both $\hat{A}\hat{H}\hat{A}^\dagger$ and $\hat{A}^\dagger\hat{H}\hat{A}$ are Hermitian.

(d) Show that any generic operator $\hat{C}$ can be expressed as a linear combination of two Hermitian operators, $\hat{R}$ and $\hat{I}$, as

$$
\hat{C} = \hat{R} + i\hat{I}. \tag{2.3}
$$

Taking the adjoint of the general operator $\hat{C}$, we find:

$$
\hat{C}^\dagger = \hat{R}^\dagger - i\hat{I}^\dagger = \hat{R} - i\hat{I}\,, \tag{2.4}
$$

since both $\hat{R}, \hat{I}$ are Hermitian operators. We may now express the operators $\hat{R}, \hat{I}$ in terms of the generic operator $\hat{C}$ and its adjoint $\hat{C}^\dagger$ as follows:

$$
R = \frac{C + C^\dagger}{2}, \quad I = \frac{C - C^\dagger}{2i}\,. \tag{2.5}
$$

Using our results from Question (b), we verify that indeed $\hat{R}$ and $\hat{I}$ are Hermitian, and thus it is possible to express any operator as $\hat{C} = \hat{R} + i\hat{I}$.

3 Commutators

During this course, we will frequently encounter commutators of two operators, both Hermitian and non-Hermitian. This information is vital towards solving a quantum mechanical problem or identifying possible symmetries of a system.

Using the definition of the commutator of two operators $[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$, show that:

- $[\hat{A}+\hat{B},\hat{C}] = [\hat{A},\hat{C}] + [\hat{B},\hat{C}]$

  $$[\hat{A}+\hat{B},\hat{C}] = (\hat{A}+\hat{B})\hat{C} - \hat{C}(\hat{A}+\hat{B}) = \hat{A}\hat{C} - \hat{C}\hat{A} + \hat{B}\hat{C} - \hat{C}\hat{B} = [\hat{A},\hat{C}] + [\hat{B},\hat{C}]$$

- $[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$

  $$[AB,C] = ABC - CAB = ABC - ACB + ACB - CAB = A[B,C] + [A,C]B$$

- $[\lambda\hat{A},\hat{B}] = \lambda[\hat{A},\hat{B}]$

  $$[\lambda\hat{A},\hat{B}] = \lambda\hat{A}\hat{B} - \hat{B}\lambda\hat{A} = \lambda(\hat{A}\hat{B} - \hat{B}\hat{A}) = \lambda[\hat{A},\hat{B}]$$

- $\big[\hat{A},[\hat{B},\hat{C}]\big] + \big[\hat{B},[\hat{C},\hat{A}]\big] + \big[\hat{C},[\hat{A},\hat{B}]\big] = 0$ (Jacobi identity)

  $$\begin{aligned}[A,[B,C]] + [B,[C,A]] + [C,[A,B]] &= A(BC - CB) - (BC - CB)A \\ &+ B(CA - AC) - (CA - AC)B \\ &+ C(AB - BA) - (AB - BA)C = 0\end{aligned}$$

where $\hat{A}$, $\hat{B}$ and $\hat{C}$ are operators (matrices), and λ is a scalar.

4 Pauli Matrices Algebra

The Pauli matrices are complex matrices that arise in the treatment of spin in quantum mechanics,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.1}$$

We will be using them frequently throughout this course, so it is essential to be comfortable with them and to understand their properties.

(a) Show that the Pauli matrices are Hermitian.

A Hermitian matrix is a (complex) square matrix that is equal to its own conjugate transpose. We need to show this for all Pauli matrices.

$$\sigma_1^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \sigma_2^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2, \quad \sigma_3^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 .$$

(b) Show that

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{1}\,, \tag{4.2}$$

where $\mathbb{1}$ is the identity matrix.

This is trivially shown by taking the square of matrices:

$$\sigma_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}\,, \tag{4.3}$$

$$\sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}\,,$$

$$\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}\,.$$

(c) Show the following identity:

$$\sigma_i \sigma_j = 1\delta_{ij} + i\epsilon_{ijk}\sigma_k, \tag{4.4}$$

where the Levi-Civita tensor is defined as:

$$\epsilon_{ijk} = \begin{cases} 1, & (ijk) \in \{(123), (231), (312)\} \\ -1, & (ijk) \in \{(132), (213), (321)\} \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

To show this result, we need to compute the product between Pauli matrices. We have already shown that if we take the square of any Pauli matrix, we get the identity matrix. The products between different Pauli matrices are:

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3$$

$$\sigma_2\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1\,,$$

$$\sigma_3\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2\,.$$

We may also easily verify that $\sigma_2\sigma_1 = -\sigma_1\sigma_2 = -i\sigma_3$, $\sigma_3\sigma_2 = -\sigma_2\sigma_3 = -i\sigma_1$ and $\sigma_1\sigma_3 = -\sigma_3\sigma_1 = -i\sigma_2$. Collecting all the results from this Question, we prove that

$$\sigma_i\sigma_j = 1\delta_{ij} + i\epsilon_{ijk}\sigma_k\,.$$

(d) Using your result from Question (c), show that:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k. \tag{4.6}$$

We expand the commutator as follows

$$[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i\,.$$

Using the result of Question (c), we find

$$[\sigma_i, \sigma_j] = 1\delta_{ij} + i\epsilon_{ijk}\sigma_k - 1\delta_{ji} - i\epsilon_{jik}\sigma_k\,.$$

Using the fact that the Kronecker delta is symmetric under index exchange, $\delta_{ij} = \delta_{ji}$, and that the Levi-Civita tensor is antisymmetric under the exchange of any pair of indices, eg $\epsilon_{ijk} = -\epsilon_{jik}$, we find

$$[\sigma_i, \sigma_j] = \cancel{1\delta_{ij}} + i\epsilon_{ijk}\sigma_k - \cancel{1\delta_{ij}} + i\epsilon_{ijk}\sigma_k = 2i\epsilon_{ijk}\sigma_k\,.$$

(e) Using your results from Question (c), show that:

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\,. \tag{4.7}$$

We expand the anticommutator as follows

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i\,.$$

Using the result of Question (c), we find

$$\{\sigma_i, \sigma_j\} = 1\delta_{ij} + i\epsilon_{ijk}\sigma_k + 1\delta_{ji} + i\epsilon_{jik}\sigma_k\,.$$

Using the fact that the Kronecker delta is symmetric under index exchange, $\delta_{ij} = \delta_{ji}$, and that the Levi-Civita tensor is antisymmetric under the exchange of any pair of indices, eg $\epsilon_{ijk} = -\epsilon_{jik}$, we find

$$\{\sigma_i, \sigma_j\} = 1\delta_{ij} + \cancel{i\epsilon_{ijk}\sigma_k} + 1\delta_{ij} - \cancel{i\epsilon_{ijk}\sigma_k} = 2\delta_{ij}\,.$$

(f) Show that for any two vectors $\vec{a}$ and $\vec{b}$, we have

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})\,, \tag{4.8}$$

where $\vec{\sigma} = \sigma_1\hat{x} + \sigma_2\hat{y} + \sigma_3\hat{z}$.

Write the LHS in terms of the matrix elements:

$$\text{LHS} = (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \sigma_i a_i \sigma_j b_j \tag{4.9}$$

Rearrange and we find

$$\text{LHS} = \sigma_i\sigma_j a_i b_j = (1\delta_{ij} + i\epsilon_{ijk}\sigma_k)a_i b_j \tag{4.10}$$

Using the properties of the delta function $\delta_{ij}a_i b_j = a_i b_i$ and the definition of the cross product $(\vec{a}\times\vec{b})_i = \epsilon_{ijk}a_j b_k$

$$\text{LHS} = \sigma_i\sigma_j a_i b_j = a_i b_i + i\sigma_k(\vec{a}\times\vec{b})_k \tag{4.11}$$

which proves the final result.

(g) Show that the eigenvalues, for a 2×2 Hermitian matrix σ with zero trace and $\sigma^2 = \mathbb{1}$, are ± 1.

We consider a general matrix σ

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

For a Hermitian matrix σ, we must have real σ_{11}, σ_{22} and $\sigma_{12} = \sigma_{21}^*$. The zero trace of σ yields $\sigma_{11} + \sigma_{22} = 0 \Rightarrow \sigma_{11} = -\sigma_{22}$. Using these results, the matrix form of σ reads

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \beta^* & -\alpha \end{pmatrix},$$

where $\alpha = \sigma_{11} \in \mathbb{R}$ and $\beta = \sigma_{12} \in \mathbb{C}$. The remaining condition $\sigma^2 = \mathbb{1}$ yields

$$|\alpha|^2 + |\beta|^2 = 1\,. \tag{4.12}$$

Let us now proceed to the diagonalization of σ. The characteristic equation for the eigenvalues λ is found

$$(-\alpha-\lambda)(\alpha-\lambda) = |\beta|^2\,,$$

which simply reduces to $\lambda^2 = |\alpha|^2 + |\beta|^2 = 1$ (using Eq. (4.12)). From this final result, we find the eigenvalues to be $\lambda_\pm = \pm 1$.

As an alternative solution, we can note that *any* Hermitian matrix M such that its square M^2 is equal to the identity must have eigenvalues equal to $+1$ or -1, even if the matrix is not 2×2 but lives in a space of higher dimension. Indeed, any Hermitian matrix is diagonalizable and has real eigenvalues. If $|v\rangle$ is an eigenvector with eigenvalue λ then $M|v\rangle = \lambda|v\rangle$ and $M^2|v\rangle = M(\lambda|v\rangle) = \lambda^2|v\rangle = |v\rangle$. This implies $\lambda^2 = 1$. Thus the eigenvalues are all either $+1$ or -1.

The trace of a Hermitian matrix is always equal to the sum of its eigenvalues. Thus for a 2×2 matrix with trace zero, the only possibility is that one eigenvalue is $+1$ and the other is -1.

(h) For each of the Pauli matrices, find its eigenvalues $\lambda_i^\pm$ and normalized eigenvectors $|v_i^\pm\rangle$, where $i = 1, 2, 3$.

All the Pauli matrices are 2×2 Hermitian matrices with zero trace and $\sigma^2 = \mathbb{1}$. Thus, using the result from the previous Question, their eigenvalues are $\lambda_i^\pm = \pm 1$, $i \in \{1, 2, 3\}$. The eigenvectors are determined as usual:

$$\left|v_1^+\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \left|v_1^-\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{4.13}$$

$$\left|v_2^+\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \quad \left|v_2^-\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \tag{4.14}$$

$$\left|v_3^+\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad \left|v_3^-\right\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{4.15}$$

(i) The expectation value of a Hermitian matrix M with respect to (w.r.t.) a normalized vector $|v\rangle$ is defined as:

$$M_v = \langle v|M|v\rangle \tag{4.16}$$

Compute:

- The expectation value of σ_3 w.r.t. its eigenvectors $\left|v_3^\pm\right\rangle$.
- The expectation value of σ_3 w.r.t. the eigenvectors of σ_2, $\left|v_2^\pm\right\rangle$.
- The expectation value of σ_2 w.r.t. its eigenvectors of $\left|v_2^\pm\right\rangle$.

The expectation value is computed in matrix form as follows:

$$M_v = \langle v|M|v\rangle = (v_1^* \quad v_2^* \cdots)\begin{pmatrix} M_{11} & M_{22} & \cdots \\ M_{21} & M_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix}$$

Thus, for the cases provided we find:

- $\left\langle v_3^+ \middle|\sigma_3\middle| v_3^+\right\rangle = 1,\ \left\langle v_3^- \middle|\sigma_3\middle| v_3^-\right\rangle = -1$
- $\left\langle v_2^+ \middle|\sigma_3\middle| v_2^+\right\rangle = 0,\ \left\langle v_2^- \middle|\sigma_3\middle| v_2^-\right\rangle = 0$
- $\left\langle v_2^+ \middle|\sigma_2\middle| v_2^+\right\rangle = 1,\ \left\langle v_2^- \middle|\sigma_2\middle| v_2^-\right\rangle = -1$

(j) Later in the course, we will see the importance of the exponential of a matrix, defined by the series:

$$e^M \equiv \sum_{n=0}^{\infty} \frac{M^n}{n!} \tag{4.17}$$

To make yourself more familiar with this object, compute:

$$\exp(i\alpha\sigma_3), \quad \exp(i\alpha\sigma_2)\,. \tag{4.18}$$

The exponential of a matrix is defined by its series. We compute $e^{i\alpha\sigma_3}$:

$$\begin{aligned} e^{i\alpha\sigma_3} &= \sum_{n=0}^{\infty} \frac{(i\alpha\sigma_3)^n}{n!} = \mathbb{1} + \frac{i\alpha\sigma_3}{1} - \frac{\alpha^2\sigma_3^2}{2!} - \frac{i\alpha^3\sigma_3^2\sigma_3}{3!} + \cdots = \\ &= \Big(1 - \frac{\alpha^2}{2!} + \cdots\Big)\mathbb{1} + i\Big(\alpha - \frac{\alpha^3}{3!}\Big)\sigma_3 = \\ &= \cos\alpha\mathbb{1} + i\sin\alpha\sigma_3 = \\ &= \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}. \end{aligned}$$

In the first to second line, we used the property $\sigma_3^2 = \mathbb{1}$, and from second to third line, we used the Taylor expansions of the sine and cosine.

$$\begin{aligned} e^{i\alpha\sigma_2} &= \sum_{n=0}^{\infty} \frac{(i\alpha\sigma_2)^n}{n!} = \mathbb{1} + \frac{i\alpha\sigma_2}{1} - \frac{\alpha^2\sigma_2^2}{2!} - \frac{i\alpha^3\sigma_2^2\sigma_2}{3!} + \cdots = \\ &= \Big(1 - \frac{\alpha^2}{2!} + \cdots\Big)\mathbb{1} + i\Big(\alpha - \frac{\alpha^3}{3!}\Big)\sigma_2 = \\ &= \cos\alpha\mathbb{1} + i\sin\alpha\sigma_2 = \\ &= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}. \end{aligned}$$

(k) Finally, we would like to understand how the exponential of a matrix acts on vectors. Apply $\exp(i\alpha\sigma_3)$ to $|v_3^+\rangle$ and decompose the result into a linear combination of $|v_3^\pm\rangle$. That is, find the coefficients $c_\pm$ in the following expansion:

$$\exp(i\alpha\sigma_3)\,|v_3^+\rangle = c_+\,|v_3^+\rangle + c_-\,|v_3^-\rangle\,. \tag{4.19}$$

Use the orthonormality of the eigenvectors for the above. Do the same with the following:

- $\exp(i\alpha\sigma_2)$ applied to $|v_2^+\rangle$ and decomposed into $|v_2^\pm\rangle$.
- $\exp(i\alpha\sigma_3)$ applied to $|v_3^-\rangle$ and decomposed into $|v_3^\pm\rangle$.
- $\exp(i\alpha\sigma_2)$ applied to $|v_3^-\rangle$ and decomposed into $|v_3^\pm\rangle$.

We use our result from the previous Question, in particular:

$$e^{i\alpha\sigma_3} = \cos\alpha\mathbb{1} + i\sin\alpha\sigma_3, \quad e^{i\alpha\sigma_2} = \cos\alpha\mathbb{1} + i\sin\alpha\sigma_2\,.$$

For the first case, we simply have:

$$e^{i\alpha\sigma_2}\,|v_2^+\rangle = \cos\alpha\,|v_2^+\rangle + i\sin\alpha\,|v_2^+\rangle = e^{i\alpha}\,|v_2^+\rangle\,.$$

For the second case:

$$e^{i\alpha\sigma_3}\left|v_3^-\right\rangle = \cos\alpha\left|v_3^-\right\rangle - i\sin\alpha\left|v_3^-\right\rangle = e^{-i\alpha}\left|v_3^-\right\rangle .$$

The third and final case requires the action of an operator on an eigenvector of a different operator. Thus, we express the target vector in the basis of σ_2:

$$\left|v_3^-\right\rangle = \frac{i}{\sqrt{2}}\Big[-\left|v_2^+\right\rangle + \left|v_2^-\right\rangle\Big] .$$

Then, the action of $e^{i\alpha\sigma_2}$ can be carried out:

$$e^{i\alpha\sigma_2}\left|v_3^-\right\rangle = e^{i\alpha\sigma_2}\frac{i}{\sqrt{2}}\Big[-\left|v_2^+\right\rangle + \left|v_2^-\right\rangle\Big] = \cdots = \sin\alpha\left|v_3^+\right\rangle + \cos\alpha\left|v_3^-\right\rangle ,$$

where in the last equality we returned back to the original basis of σ_3. You may verify that in all of the above cases, the resulting states remain normalized.

(l) Show that if $\hat{A}$ and $\hat{B}$ are two operators such that $[\hat{A},\hat{B}] = c$ then $e^{x\hat{A}}\hat{B}e^{-x\hat{A}} = B + cx$.

The question can be solved by two different methods. *Solution 1.* First, note that if $[\hat{A},\hat{B}] = c$, then $[\hat{A}^2,\hat{B}] = \hat{A}[\hat{A},\hat{B}]+[\hat{A},\hat{B}]\hat{A} = 2c\hat{A}$, $[\hat{A}^3,\hat{B}] = \hat{A}[\hat{A}^2,\hat{B}]+[\hat{A},\hat{B}]\hat{A}^2 = 3c\hat{A}^2$.. In general, the relation $[\hat{A}^n,\hat{B}] = \hat{A}[\hat{A}^{n-1},\hat{B}] + [\hat{A},\hat{B}]\hat{A}^{n-1} = \hat{A}[\hat{A}^{n-1},\hat{B}] + c\hat{A}^{n-1}$ can be used to prove by induction that $[\hat{A}^n,\hat{B}] = nc\hat{A}^{n-1}$. Using this result we can see that

$$[e^{x\hat{A}},\hat{B}] = \sum_{n=0}^{\infty}\frac{x^n}{n!}[\hat{A}^n,\hat{B}] = \sum_{n=1}^{\infty}\frac{cnx^n\hat{A}^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{cx^n\hat{A}^{n-1}}{(n-1)!} = x\sum_{n=1}^{\infty}\frac{cx^{n-1}\hat{A}^{n-1}}{(n-1)!} = cxe^{x\hat{A}} . \tag{4.20}$$

Then:

$$e^{x\hat{A}}\hat{B}e^{-x\hat{A}} = \hat{B}e^{x\hat{A}}e^{-x\hat{A}} + [e^{x\hat{A}},\hat{B}]e^{-x\hat{A}} = \hat{B} + cxe^{x\hat{A}}e^{-x\hat{A}} = \hat{B} + cx . \tag{4.21}$$

Note that $e^{x\hat{A}}e^{-x\hat{A}} = \hat{1}$, the identity operator, is valid for any scalar x and any operator $\hat{A}$. The simplest way to see this is by noting that the operator $\hat{A}$ commutes with itself and with all of its powers $[\hat{A},\hat{A}^k] = 0$. So the expansion of the product of exponentials $e^{x\hat{A}}e^{-x\hat{A}}$ proceeds essentially as for standard real numbers, as for all terms the ordering of operators is unimportant. Then, exactly as for real numbers we get that in the expansion of $e^{x\hat{A}}e^{-x\hat{A}}$ all powers of x have a vanishing coefficient and the result is equal to 1 (the identity operator).

This can be verified using

$$e^{x\hat{A}}e^{-x\hat{A}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m x^{n+m}\hat{A}^{n+m}}{n!m!} = \sum_{k=0}^{\infty}\frac{x^k\hat{A}^k}{k!}\sum_{m=0}^{k}\frac{(-1)^m k!}{(k-m)!m!} \tag{4.22}$$

where in the last expression we grouped terms with the same value of $n+m$. Using that $\sum_{m=0}^{k}(-1)^m k!/((k-m)!m!) = (1-1)^k$ for $k \geq 1$ and $\sum_{m=0}^{k}(-1)^m k!/((k-m)!m!) = 1$ for $k = 0$ we recover the result.

Note however that $e^{x\hat{A}}\hat{B}e^{-x\hat{A}} \neq \hat{B}$, a result which follows from the fact that $\hat{A}$ and $\hat{B}$ do *not* commute.

Solution 2. A second solution can be derived by noting that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x\hat{A}}\hat{B}e^{-x\hat{A}}\right) = e^{x\hat{A}}(\hat{A}\hat{B} - \hat{B}\hat{A})e^{-x\hat{A}} = ce^{x\hat{A}}e^{-x\hat{A}} = c\,. \tag{4.23}$$

Integrating over time with the initial condition $e^{x\hat{A}}\hat{B}e^{-x\hat{A}} = \hat{B}$ at $x = 0$ shows that $e^{x\hat{A}}\hat{B}e^{-x\hat{A}} = \hat{B} + cx$.

This second solution is closely analogue, from an algebraic point of view, to the solution of Heisenberg equations of motion, which will be described in the course.